Paraxial beams of spinning light

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ABSTRACT

A simple derivation is given for the known result that the component of total angular momentum along the propagation direction of a general paraxial beam can be separated into orbital and spin parts. Comments are made about orbital angular momentum and wave dislocations, and about how the orbital and spin angular momentum can be changed by propagation in a refracting medium.

1. ANGULAR MOMENTUM DECOMPOSITION

It is now well established by theory and experiment that beams of light can carry both orbital and spin angular momentum[1-4], at least in the paraxial approximation[5]. In spite of this widespread understanding, it seems worthwhile to present the argument in its most general form, and that is my main purpose here - without claiming originality for the central ideas. The aims are to represent the light in terms of general transverse fields (i.e. not necessarily as a superosition of Gauss-Laguerre beams), to allow the polarization to vary with position, to introduce paraxiality in the most direct way, and to write the result in a way that resembles quantum mechanics, so that the separation into the two kinds of angular momentum is compelling. In addition, I will make some remarks about the significance of the two components, and how they can be changed by the medium through which the light is propagating.

Let monochromatic light with frequency ω (and wavenumber $k = \omega/c$) travel in vacuum in the paraxial direction z (unit vector $\mathbf{e}_Z$). A precise specification of z will be given later. Let positions in the transverse plane be denoted by $\mathbf{r}$, let the transverse part of a general vector be denoted by $\mathbf{V}$, and let the full three-dimensional counterparts of these quantities be denoted by the suffix "tot". Thus

$$\mathbf{r} = (\mathbf{x},\mathbf{y}), \quad \mathbf{V} = (\mathbf{A}_x,\mathbf{A}_y), \quad \mathbf{r}_{\text{tot}} = (\mathbf{x},\mathbf{y},\mathbf{z}), \quad \mathbf{V}_{\text{tot}} = (\mathbf{A}_x,\mathbf{A}_y,\mathbf{A}_z) \tag{1}$$

We will calculate the z component of the total angular momentum per photon in unit length of a transverse slice of the beam about the axis $\mathbf{r} = 0$, and denote this quantity by J_Z. We use standard notation for the complex electromagnetic fields, and the well known results[6] that the angular momentum density is

$$\mathrm{Re}\,\mathbf{r}\times\left(\mathbf{D}^*_{\text{tot}}\times\mathbf{B}_{\text{tot}}\right) \tag{2}$$

and the energy density is

$$\mathrm{Re}\,\tfrac{1}{2}\left(\mathbf{E}^*_{\text{tot}}\cdot\mathbf{D}_{\text{tot}}+\mathbf{B}^*_{\text{tot}}\cdot\mathbf{H}_{\text{tot}}\right) \tag{3}$$

Then

$$J_z = \hbar\omega\frac{\mathrm{Re}\iint dxdy\,\mathbf{r}\times\left(\mathbf{D}^*_{\text{tot}}\times\mathbf{B}_{\text{tot}}\right)\cdot\mathbf{e}_z}{\mathrm{Re}\,\frac{1}{2}\iint dxdy\left(\mathbf{E}^*_{\text{tot}}\cdot\mathbf{D}_{\text{tot}}+\mathbf{B}^*_{\text{tot}}\cdot\mathbf{H}_{\text{tot}}\right)} \tag{4}$$

Maxwell's equations and the constitutive relations can be used to eliminate all fields except $\mathbf{E}_{\text{tot}}$, with the result

$$J_z = \hbar \frac{\mathrm{Re}(-\mathrm{i})\iint dxdy\, \mathbf{r}\times\left(\mathbf{E}^*_{\mathrm{tot}}\times(\nabla\times\mathbf{E}_{\mathrm{tot}})\right)\cdot\mathbf{e}_z}{\mathrm{Re}\,\frac{1}{2}\iint dxdy\left(\mathbf{E}^*_{\mathrm{tot}}\cdot\mathbf{E}_{\mathrm{tot}}+\frac{1}{k^2}(\nabla\times\mathbf{E}^*_{\mathrm{tot}})\cdot(\nabla\times\mathbf{E}_{\mathrm{tot}})\right)} \tag{5}$$

In this exact formula, all components of $\mathbf{E}_{\mathrm{tot}}$ can vary with x, y and z.

Now we invoke the paraxial approximation and then $\nabla\cdot\mathbf{E}_{\mathrm{tot}}=0$ to eliminate all z derivatives and z components to order $1/k$:

$$\partial_z\mathbf{E}_{\mathrm{tot}} \approx \mathrm{i}\,k\mathbf{E}_{\mathrm{tot}}, \quad E_z \approx \frac{\mathrm{i}}{k}\nabla\cdot\mathbf{E} \tag{6}$$

Thus, in the denominator in (5),

$$\mathbf{E}^*_{\mathrm{tot}}\cdot\mathbf{E}_{\mathrm{tot}} \approx \frac{1}{k^2}(\nabla\times\mathbf{E}^*_{\mathrm{tot}})\cdot(\nabla\times\mathbf{E}_{\mathrm{tot}}) \approx \mathbf{E}^*\cdot\mathbf{E} \tag{7}$$

The numerator can be similarly simplified using (6) and also integration by parts with respect to x and y, leading after a little algebra to

$$J_z \approx \frac{\mathrm{Re}(-\mathrm{i}\hbar)\iint dxdy\left\{\mathbf{E}^*\cdot\left[\mathbf{e}_z\cdot\left(\mathbf{r}\times(-\mathrm{i}\nabla)\right)\right]\mathbf{E}+\mathbf{e}_z\cdot\mathbf{E}^*\times\mathbf{E}\right\}}{\mathrm{Re}\iint dxdy\,\mathbf{E}^*\cdot\mathbf{E}} \tag{8}$$

It is well known[5] that a similar expression, with all vectors replaced by their three-dimensional counterparts, can be obtained exactly - that is, without the paraxial approximation - if the quantity being calculated is the total angular momentum in the whole three-dimensional beam; but this depends on integration with respect to z, an option unavailable in our calculation for a transverse slice.

In the final step, we introduce a basis of circular polarizations, replacing $\mathbf{E}$ by the column vector

$$|\psi\rangle \equiv \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}E_x-\mathrm{i}\,E_y\\ E_x+\mathrm{i}\,E_y\end{pmatrix}. \tag{9}$$

Then we define scalar products as

$$\langle\phi|\psi\rangle \equiv \iint dxdy\begin{pmatrix}\phi^*_+(\mathbf{r}) & \phi^*_-(\mathbf{r})\end{pmatrix}\begin{pmatrix}\psi_+(\mathbf{r})\\ \psi_-(\mathbf{r})\end{pmatrix}, \tag{10}$$

and recognize

$$\mathbf{p} = -\mathrm{i}\hbar\nabla \tag{11}$$

as the momentum operator, and

$$l_z = \mathbf{r}\times\mathbf{p}\cdot\mathbf{e}_z, \quad s_z = \hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{12}$$

as the operators for the z components of the orbital angular momentum, and the spin angular momentum of a spin-1 particle. Thus (8) becomes the well known expression

$$J_z \approx \frac{\langle\psi|l_z|\psi\rangle + \langle\psi|s_z|\psi\rangle}{\langle\psi|\psi\rangle} \equiv L_z + S_z \tag{13}$$

This is valid in the paraxial approximation, that is when quantities of order $1/k$ are neglected. Henceforth we assume without loss of generality that the beam is normalized with $\langle\psi|\psi\rangle = 1$.

2. REMARKS ON SPIN AND ORBITAL PARTS OF THE ANGULAR MOMENTUM

It might be thought that the difference between the spin and orbital contributions S_z and L_z in (13) is that S_z is independent of the axis $\mathbf{r} = 0$ about which J_z is calculated, whereas L_z depends on this axis. But this can be wrong, as the following argument shows. Under a shift of origin, $\mathbf{r} \to \mathbf{r}_0 + \mathbf{r}$, the orbital angular momentum shifts too:

$$L_z \to L_z + \mathbf{e}_z \cdot \mathbf{r}_0 \times \mathbf{P}, \quad \text{where } \mathbf{P} \equiv \langle\psi|\mathbf{p}|\psi\rangle. \tag{14}$$

Here **P** is the transverse current in the beam. If **P** vanishes, the shift $\mathbf{r} \to \mathbf{r}_0 + \mathbf{r}$ leaves L_z unaltered. But for a paraxial beam **P** will always be small compared with the current along the paraxial direction z. Therefore it is possible and indeed reasonable to stipulate z as the direction for which **P** is *exactly* zero. With paraxiality thus defined, *both* components of J_z are invariant under a shift of axis, so this invariance cannot be used to define S.

The true distinction between orbital and spin angular momentum is of course that L_z depends on the spatial structure of the beam irrespective of polarization, whereas S_z depends on the state of polarization of the beam. There are subtleties concealed here too. To appreciate these, note that, from (9), (10), (12) and (13),

$$S_z = \hbar\langle\psi|\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}|\psi\rangle = \hbar\iint dxdy\left(|\psi_+(\mathbf{r})|^2 - |\psi_-(\mathbf{r})|^2\right), \tag{15}$$

so that $S_z = \hbar$ for right-circularly polarized light ($\psi_- = 0$), and $S_z = -\hbar$ for left-circularly polarized light ($\psi_+ = 0$), as expected for spin-1 particles. For linearly polarized light ($|\psi_+| = |\psi_-|$), $S_z = 0$. A complete description of the polarization state involves the other two Pauli matrices, and suggests defining

$$S_x = \hbar\langle\psi|\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}|\psi\rangle, \quad S_y = \hbar\langle\psi|\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}|\psi\rangle. \tag{16}$$

But S_x, S_y and S_z must not be regarded as the three components of a spin angular momentum vector of the light, although they give a complete description of the state of polarization of the beam. One reason is that they have the wrong commutation relations, e.g. $[S_x, S_y] = 2i\hbar S_z$ (rather than $i\hbar S_z$ as would be appropriate to a spin-1 particle); another is that $S_x^2 + S_y^2 + S_y^2 = 3\hbar^2$ (rather than $2\hbar^2$ as would be appropriate to a spin-1 particle).

One of several correct representations of spin-1 matrices is that implied by the three-dimensional version of the last term in (8), namely

$$\mathrm{Re}\left(-i\hbar\mathbf{E}^*_{tot} \times \mathbf{E}_{tot}\right) = \mathbf{E}^*_{tot} \cdot \mathbf{S}_{tot} \cdot \mathbf{E}_{tot}, \tag{17}$$

where

$$\mathbf{S}_{tot} = -i\hbar\left\{\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix}, \begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}\right\} \tag{18}$$

Strangely, however, these spin-1 matrices, unlike those in (15) and (16), do not give a complete description of the state of polarization of the beam: for any transverse wave, whatever its polarization, the transverse components of the expectation value of S_{tot} vanish, leaving only the single (real) z component nonvanishing, and unable to discriminate, for example, between the different possible states of linear polarization. There is another well known 'paradox' associated with the full three-dimensional angular momentum of a transverse light wave (e.g. a plane wave): the exact equation (2) implies that all components of total angular momentum vanish for such a wave, whereas (13) predicts a nonvanishing value for J_Z for a circularly polarized transverse wave. The resolution lies in the fact that (13) applies to a beam whose lateral extent is effectively finite, which can therefore not be exactly transverse; in the derivation of (13), the longitudinal component (approximated by (6)) contributed through the integration by parts.

Turning now to the orbital angular momentum L_Z, we note the common opinion[7] that this quantity is associated with optical dislocation lines (phase singularities)[8-11]. In general, however, there is no such association, as the following argument shows.

Since L_Z is associated with the spatial variation of the beam, it suffices to take for $|\psi\rangle$ a scalar wavefunction $\psi(\mathbf{r},z)$, satisfying the paraxial wave equation. We let ψ be a 'spiral beam' [12, 13] with waist size w; In the focal plane $z = 0$ this has the form

$$\psi(\mathbf{r},0) = f(\zeta)\exp\{-r^2/2w^2\}, \quad \zeta \equiv x + i\,y, \quad r \equiv \sqrt{(x^2+y^2)}. \tag{19}$$

Away from the focal plane, ψ preserves this form while expanding and rotating. A short calculation gives

$$L_z = \hbar\frac{\mathrm{Re}\iint dxdy\exp(-r^2/w^2)f(\zeta)^*\zeta\partial_\zeta f(\zeta)}{\mathrm{Re}\iint dxdy\exp(-r^2/w^2)|f(\zeta)|^2}. \tag{20}$$

If $f(\zeta)$ is an Mth order polynomial, that is

$$f(\zeta) = \sum_{m=0}^{M} f_m\zeta^m, \tag{21}$$

then

$$L_z = \hbar\frac{\sum_{m=1}^{M} m\,m!|f_m|^2 w^{2m}}{\sum_{m=0}^{M} m!|f_m|^2 w^{2m}} \equiv \langle m\rangle\hbar. \tag{22}$$

Now, dislocations are the zeros of ψ, that is the zeros of $f(\zeta)$, and by the fundamental theorem of algebra there are exactly M of them for the spiral beam with f given by (21). Moreover, they all have the same sign, so the total dislocation strength[11] is also M. In the special case of a pure Gaussian beam, where only the term $m = M$ in the series (21) is nonzero, $\langle m\rangle = M$ and there is a sense in which it is correct to ascribe the (quantized) orbital angular momentum to the Mth order (screw) dislocation. But in all other cases, $<m> \neq M$ and the association between L_Z and dislocations is false. The easy way to see that this must be so is to note that since dislocations are zeros they have zero weight in the integral (20) for L_Z.

To avoid confusion, I should remark that there is an association between a dislocation line and the torque exerted on a small particle[2-4], arising from the vortex structure close to the line. This very interesting effect concerns the angular momentum density in the vicinity of the particle, rather than the integrated quantity L_Z considered here.

3. BEAM PROPAGATION IN A REFRACTING MEDIUM

If the beam propagates paraxially in a homogenous and isotropic medium, no torque can be exerted on it, and L_Z and S_Z are separately conserved. However, an inhomogenous medium can exert a torque that changes L_Z, and an anisotropic medium can exert a torque that changes S_Z. To quantify these effects, let the medium have dielectric tensor

$$\varepsilon(\mathbf{r}) = \varepsilon_0\left(1 + 2\mathbf{n}(\mathbf{r})\right). \tag{23}$$

Here **n** is a transverse refractive index tensor, hermitean because the medium is assumed transparent, whose components

$$\mathbf{n}(\mathbf{r}) = \begin{pmatrix} n_{xx}(\mathbf{r}) & n_{xy}(\mathbf{r}) \\ n_{xy}(\mathbf{r}) & n_{zz}(\mathbf{r}) \end{pmatrix} \tag{24}$$

are all much less than unity since the propagation is paraxial. In this case contributions from **n** to J_Z are of higher than paraxial order, and the argument leading from (2) and (3) to the final formula (13) is unaffected by the anisotropy and variations of the medium.

However, propagation of the angular momentum is affected, in a manner determined by the paraxial wave equation for **E**. From Maxwell's equations, this is

$$ik\partial_z \mathbf{E} = -\tfrac{1}{2}\left(\partial_x^2 + \partial_x^2\right)\mathbf{E} - k^2 \mathbf{n}(\mathbf{r})\cdot\mathbf{E} \ . \tag{25}$$

Transforming to the basis (9) of circular polarizations gives the 'Schrödinger lookalike' equation

$$\mathrm{i}k\partial_z|\psi\rangle = \mathbf{H}|\psi\rangle \ . \tag{26}$$

Here the 'hamiltonian' **H** (now setting $\hbar = 1$ for simplicity of notation) is

$$\mathbf{H} = \tfrac{1}{2}\mathbf{p}\cdot\mathbf{p} - k^2\left\{W_0(\mathbf{r}) + \mathbf{W}(\mathbf{r})\cdot\sigma\right\}, \tag{27}$$

in which σ is the (three-dimensional) vector of Pauli matrices and $W_0(\mathbf{r})$ and the three-dimensional vector $\mathbf{W}(\mathbf{r})$ are (invoking the hermiticity of **n**)

$$\begin{gathered} W_0 = n_{xx} + n_{yy}, \\ \mathbf{W} = \left\{W_x, W_y, W_z\right\} = \left\{n_{xx} - n_{yy}, 2\,\mathrm{Re}\,n_{xy}, -2\,\mathrm{Im}\,n_{xy}\right\} \end{gathered} \tag{28}$$

Note that W_0 depends on the isotropic part of the refractive index of the medium, W_X and W_Y to its birefringence, and W_Z to its optical activity (chirality).

The rate of change in S_Z is easily shown to be

$$\partial_z S_z = 2k\langle\psi|\mathbf{W}\times\sigma\cdot\mathbf{e}_z|\psi\rangle$$
$$= 2k\langle\psi|\begin{pmatrix} 0 & -\mathrm{i}(n_{xx}-n_{yy})-2\,\mathrm{Re}\,n_{xy} \\ \mathrm{i}(n_{xx}-n_{yy})-2\,\mathrm{Re}\,n_{xy} & 0 \end{pmatrix}|\psi\rangle \tag{29}$$

Thus the spin angular momentum can be changed by the birefringence of the medium but not by its optical activity.

Similarly, the rate of change of L_z is determined by the commutator in

$$\partial_z L_z = -\mathrm{i}\,k\langle\psi|[W_0(\mathbf{r})+\mathbf{W}(\mathbf{r})\cdot\sigma,\ l_z]|\psi\rangle. \tag{30}$$

Since, from the definition (12),

$$l_z = -\mathrm{i}\,\partial_\phi \tag{31}$$

where ϕ is the polar angle of $\mathbf{r}$, we recover the obvious result that the orbital angular momentum can be changed only by propagation in a medium that is not azimuthally symmetric, regardless of whether the asymmetry is in the isotropic part W_0 of the medium or its anisotropic (birefringent or chiral) part $\mathbf{W}$.

During any such changes of its angular momentum, the beam will, of course, exert equal and opposite reactive torques on the medium.

4. ACKNOWLEDGEMENTS

I thank Professor M.S. Soskin for generous hospitality at the conference on Singular Optics, and Professors N.R. Heckenberg, A. Alexeyev and C. Alexeyev for useful discussions.

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