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# Smooth and oscillatory geometric phase corrections for driven spins

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**Abstract**

For a quantum spin driven cyclically by a slowly-rotated magnetic field, geometric phases are well understood. If the cycle takes a long time $T$, the leading-order (dynamical) phase is proportional to $T$ and the geometric phase is the contribution independent of $T$. The dynamical and geometric phases are the first two terms of a series in slowness $1/T$. Here it is shown with an exactly solvable example that the corrections are of two types: smooth, proportional to powers of slowness, and oscillatory: essential singularities in $1/T$, in the form of trigonometric functions of $T$ divided by powers of $T$. The calculations are elementary and therefore suitable for presentation in graduate quantum theory courses.

Keywords: adiabatic, oscillations, quantum

(Some figures may appear in colour only in the online journal)

## 1. Introduction

Quantum states can be driven by time-dependent Hamiltonians $H(t)$ that are cyclic: $H(t)$ returns to its original form after time $T$. If $T$ is large—that is, if the driving is slow, with slowness parameter $1/T$—the adiabatic theorem [1–3] guarantees that an initial state occupying one of the instantaneous eigenstates clings closely to that eigenstate and so almost returns to the initial state after time $T$, apart from a phase. 'Almost' means more closely for larger $T$. The phase is large: its dominant contribution, the dynamical phase, is proportional to

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$T$. The first correction is independent of $T$; this is the geometric phase [2, 4–6], involving only the sequence of Hamiltonians $H(t)$ but not the rate at which it is cycled.

My concern here is with the nature of the corrections: the contributions to the accumulated phase, beyond dynamical and geometric, that vanish as $T \to \infty$. There are a number of studies of beyond-adiabatic geometric phase corrections, and more general adiabatic approximation schemes [3, 7–11]. My aim here is to point out that there are two distinct types of corrections. The first kind are smooth: proportional to powers of slowness $1/T$. The second kind are nonanalytic in $1/T$, for example oscillatory. We will demonstrate the two kinds of corrections in section 2, using the much-studied and exactly solvable example [2, 11–14] of a spin $1/2$ quantum particle driven by a rotating magnetic field. This has at least two pedagogical advantages, making the analysis reported here suitable for graduate quantum theory courses.

First, all the calculations are elementary: only convergent series are involved. By contrast, other geometric phase related corrections, described in the concluding section 4, involve divergent series. The present study is an easy way to demonstrate the separation of the kinds of geometric phase corrections, avoiding sophisticated asymptotics.

The second advantage is that it illustrates how new observations can still be made in problems that are almost a century old.

Section 3 displays a numerical calculation of the phase corrections, illustrating the separation between smooth and oscillatory contributions. The concluding section 4 contrasts the case considered here with two others: the more general situation where the adiabatic series diverges, and the series of reaction forces on the dynamics of the slow driving parameters. Section 4 also includes a discussion of the Aharonov–Anandan (AA) phase [15], in which it is the states, rather than the Hamiltonians, that are cycled exactly. Appendix A gives details of the solution of the Schrödinger equation, and appendix B makes explicit the connections with the AA phase.

We need to recall basic geometric phase theory [6]. The time-dependent Schrödinger equation and cycled Hamiltonian are, in units where $\hbar = 1$,

$$i\partial_t|\psi(t)\rangle = H(t)|\psi(t)\rangle,\;\; H(t+T) = H(t). \tag{1.1}$$

The energies and eigenstates of the instantaneously frozen Hamiltonian are

$$H(t)|n(t)\rangle = E_n(t)|n(t)\rangle,\;\; |n(t+T)\rangle = |n(t)\rangle, \tag{1.2}$$

indicating that the arbitrary phases of the eigenstates have been chosen to make the states return exactly at $t = T$, in order to make the geometric phase emerge in its simplest form [6].

For the initial state, we choose one of the eigenstates:

$$|\psi(0)\rangle = |n(0)\rangle\;. \tag{1.3}$$

At $t = T$, the driven state has almost returned, up to a phase. The phase is predominantly dynamical: the integral of the instantaneous energy of the state $|n(t)\rangle$. We define the remaining phase $\gamma_n(T)$ exactly [16], in terms of the phase of the overlap of the final and initial states, after subtraction of the dynamical phase:

$$\arg[\langle n(0)|\psi(T)\rangle] = -\int_0^T \mathrm{d}t\; E_n(t) + \gamma_n(T). \tag{1.4}$$

(The modulus of the overlap gives the probalility that there has been a nonadiabatic transition away from the eigenstate $|n(t)\rangle$.)

We are interested in the $T$ dependence of $\gamma_n(T)$. In the adiabatic (long-time) limit of infinitely slow driving, this is simply the geometric phase, for which standard arguments [6] give

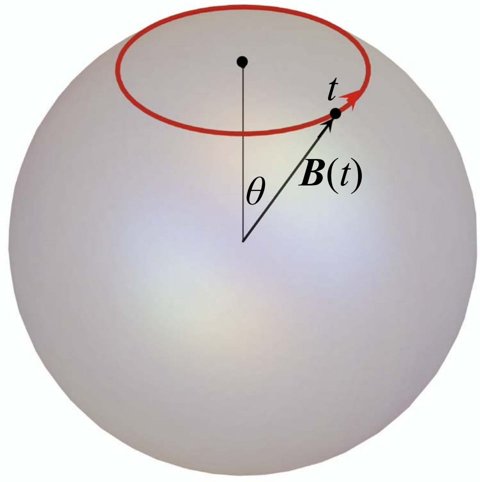


**Figure 1.** Rotating magnetic driving field.

$$\gamma_n(\infty) = \gamma_{n,\mathrm{geom}} = -\mathrm{Im}\left[\int_0^T \mathrm{d}t\ \langle n(t)|\partial_t n(t)\rangle\right]. \tag{1.5}$$

This can be expressed as the integral of a 1-form around the cycle, or, by Stokes's theorem, the integral of a 2-form ('curvature') over any surface spanning the cycle. These aspects, briefly mentioned in section 4, are not relevant in the present context, which is to investigate the correction to the dynamical plus geometric phase, defined as

$$\Delta\gamma_n(T) = \gamma_n(T) - \gamma_{n,\mathrm{geom}}. \tag{1.6}$$

Of course the phase is defined only mod($2\pi$), because the physically significant quantity is not the phase but the phase factor (see [6, 17]). During the calculations to follow, irrelevant multiples of $2\pi$ will be ignored.

## 2. Application to driven spin

This familiar exactly solvable 'Rabi' Hamiltonian [2, 12, 13] represents a spin $1/2$ particle driven by a magnetic field rotating around a cone (figure 1):

$$H(t) = B(t)\cdot\sigma = \begin{pmatrix} \cos\theta & \exp\left(-i2\pi\frac{t}{T}\right)\sin\theta \\ \exp\left(i2\pi\frac{t}{T}\right)\sin\theta & -\cos\theta \end{pmatrix}. \tag{2.1}$$

Here the driving field is

$$B(t) = \left\{\sin\theta\cos\left(2\pi\frac{t}{T}\right),\ \sin\theta\sin\left(2\pi\frac{t}{T}\right),\ \cos\theta\right\}, \tag{2.2}$$

and $\boldsymbol{\sigma}$ is the vector of Pauli matrices:

$$\sigma = \left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\}. \tag{2.3}$$

In physical units, the Hamiltonian would include the gyromagnetic ratio and the magnitude of the field, as well as $\hbar$, but these quantities can be eliminated by scaling $t$ and $T$, leaving (2.1).

The instantaneous eigenenergies (independent of time in this case) are

$$E_{\pm} = \pm 1, \tag{2.4}$$

and the instantaneous eigenvectors are

$$|+(t)\rangle = \begin{pmatrix} \exp\left(-i2\pi\frac{t}{T}\right)\cos\frac{1}{2}\theta \\ \sin\frac{1}{2}\theta \end{pmatrix}, |-(t)\rangle = \begin{pmatrix} \exp\left(-i2\pi\frac{t}{T}\right)\sin\frac{1}{2}\theta \\ -\cos\frac{1}{2}\theta \end{pmatrix}. \tag{2.5}$$

The phase factors involving time have been chosen to make the eigenvectors return exactly at the end of the cycle $t = T$. A simple calculation from (1.5) gives the standard geometric phases for the two states, in terms of half the solid angle subtended by the magnetic field cycle [6]:

$$\gamma_{\pm.\mathrm{geom}} = \mp\pi(1 - \cos\theta) \tag{2.6}$$

In this example, the physically irrelevant $2\pi$ phase ambiguity has a simple geometrical origin: the two complementary caps on the $\boldsymbol{B}$ sphere, with solid angles differing by the $4\pi$ solid angle of the sphere.

We choose to start from the + state:

$$|\psi(0) = |+(0) = \begin{pmatrix} \cos\dfrac{1}{2}\theta \\ \sin\dfrac{1}{2}\theta \end{pmatrix} \tag{2.7}$$

As explained in appendix A, and as can be confirmed by direct substitution, the exact solution of (1) with (2.1) and the initial state (2.7) gives the evolving state [11, 12]

$$|\psi(t) = \begin{pmatrix} \exp\left(-i\pi\dfrac{t}{T}\right)\cos\dfrac{1}{2}\theta\left[\cos\left(\dfrac{t}{T}R(T)\right) + i\dfrac{(\pi - T)\sin\left(\dfrac{t}{T}R(T)\right)}{R(T)}\right] \\ \exp\left(i\pi\dfrac{t}{T}\right)\sin\dfrac{1}{2}\theta\left[\cos\left(\dfrac{t}{T}R(T)\right) - i\dfrac{(\pi + T)\sin\left(\dfrac{t}{T}R(T)\right)}{R(T)}\right] \end{pmatrix} \tag{2.8}$$

in which

$$R(T) = \sqrt{T^2 - 2\pi T\cos\theta + \pi^2}\,. \tag{2.9}$$

The significance of $R(T)$ is explained in appendix A (see also equation (140) in [11]). The overlap in (1.4) is

$$\langle +(0)|\psi(T)\rangle = -\cos R(T) + i\frac{(T - \pi\cos\theta)}{R(T)}\sin R(T). \tag{2.10}$$

In terms of $R(T)$, and also $F(T)$, defined by

$$\frac{T - \pi\cos\theta}{R(T)} = 1 - F(T)\,, \tag{2.11}$$

the overlap is

$$\langle +(0)|\psi(T)\rangle = -\exp(-iR(T)) - iF(T)\sin R(T) = \\ -\exp(-iR(T))\left(1 - F(T)\sin^2 R(T) + \frac{1}{2}iF(T)\sin(2R(T))\right) \tag{2.12}$$

from which the remaining phase in (1.4) is

$$\gamma_+(T) = T - \pi - R(T) + \arctan\left(\frac{F(T)\sin(2R(T))}{2(1 - F(T)\sin^2 R(T))}\right) \tag{2.13}$$

This formula appears as equation (173) in a comprehensive study [11] of adiabatic correction schemes.

We need the large $T$ behaviour of the functions $R(T)$ and $F(T)$; both involve convergent series. For $R(T)$, the series is

$$R(T) = T - \pi\cos\theta + \sum_{m=1}^{\infty}\frac{R_m}{T^m}\Bigg), \tag{2.14}$$

where the first terms are

$$R_1 = \frac{\pi^2}{2}\sin^2\theta,\ \ R_2 = \frac{\pi^3}{2}\sin^2\theta\cos\theta,\ R_3 = \frac{\pi^4}{16}\sin^2\theta(3 + 5\cos 2\theta)\ , \tag{2.15}$$

and for $F(T)$, the series is

$$F(T) = \sum_{m=2}^{\infty}\frac{F_m}{T^m}, \tag{2.16}$$

where the first terms are

$$F_2 = \frac{\pi^2}{2}\sin^2\theta,\ F_3 {=} \pi^3\sin^2\theta\cos\theta,\ F_4 = \frac{3\pi^4}{16}\sin^2\theta\quad(3 + 5\cos 2\theta). \tag{2.17}$$

The fact that the series for $R(t)$ and $F(T)$ converge is central to the simple separation of smooth and oscillatory corrections for the Hamiltonian (2.1). Usually, adiabatic series (i.e. series in powers of slowness $1/T$) do not converge, as discussed in the concluding section 4. The present paper concerns spin 1/2, but the corresponding series would converge for any spin, provided its driving magnetic field is uniformly rotated.

Using these expansions, the two parts of (2.13)—without and with the arctangent—give the contributions to the remaining phase: smooth (powers of $1/T$), and essentially singular (nonalytic in $1/T$). For the spin system considered here, the essentially singular terms are oscillations: trigonometric functions of multiples of $T$. The separation (2.13) is clean: it can be confirmed that all terms in the expansion of $R(T)$ are smooth, and all terms in the expansion of the arctangent are oscillatory. Explicitly, the first few terms are

$$\Delta\gamma_+(T) = -\frac{\pi^2}{2T}\sin^2\theta - \frac{\pi^3}{2T^2}\sin^2\theta\cos\theta - \frac{\pi^4}{16T^3}\sin^2\theta\ \ (3 + 5\cos 2\theta) \\ + \frac{\pi^2}{4T^2}\sin^2\theta\sin(2\tau) + \frac{\pi^3}{8T^3}\sin^2\theta\ (4\cos\theta\ \sin(2\tau) + \pi\sin^2\theta\ \cos(2\tau)) + \cdots \tag{2.18}$$

in which

$$\tau = T - \pi\cos\theta \tag{2.19}$$

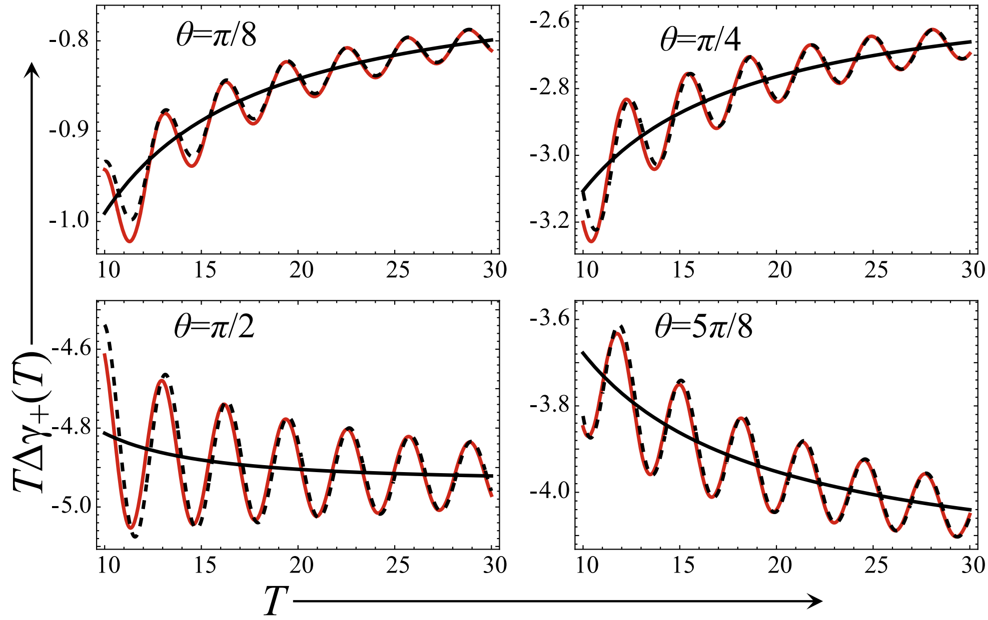


**Figure 2.** Geometric phase corrections for the indicated values of $\theta$. Red curve: difference (1.6) between the remaining phase (1.4) and the geometric phase; black curve: first three terms (2.18) in the series for the smooth phase contribution; dashed curve: after including the first two oscillatory terms (the terms involving $\tau$ in (2.18)). All values are multiplied by $T$ to accentuate the oscillations (see (2.19).

The leading-order correction to the geometric phase is smooth, and of order $1/T$. The leading-order oscillatory contribution appears at order $1/T^2$.

## 3. Numerical illustration

To display the comparison between the exact phase correction (lhs of (2.18)) and the first few corrections (rhs of (2.18)), it is convenient to multiply by $T$, that is, to study $T\Delta\gamma_+(T)$. This has the advantage of making the leading-order smooth correction constant, corresponding to the asymptote

$$\lim_{T\to\infty} T\Delta_+\gamma(T) = -\frac{\pi^2}{2}\sin^2\theta, \tag{3.1}$$

and accentuating the subsequent corrections, oscillatory as well as smooth.

Figure 2 shows the comparison for several magnetic cone angles $\theta$. Immediately evident is how the oscillatory contributions decorate the smooth contributions, and how the accuracy of the approximation increases with $T$.

To emphasise that figure 2 shows small corrections, table 1 gives some corresponding much larger numbers for $T$ times the geometrical and dynamical phases.

Also interesting is to show the track of the expectation of the spin unit vector

$$\boldsymbol{s}_+(t) = \langle\psi(t)|\boldsymbol{\sigma}|\psi(t)\rangle \tag{3.2}$$

Figure 3 shows the tracks for increasing values of $T$, illustrating how the spin not only precesses along with the driving field but also nutates, faster with increasing $T$. Nutation has

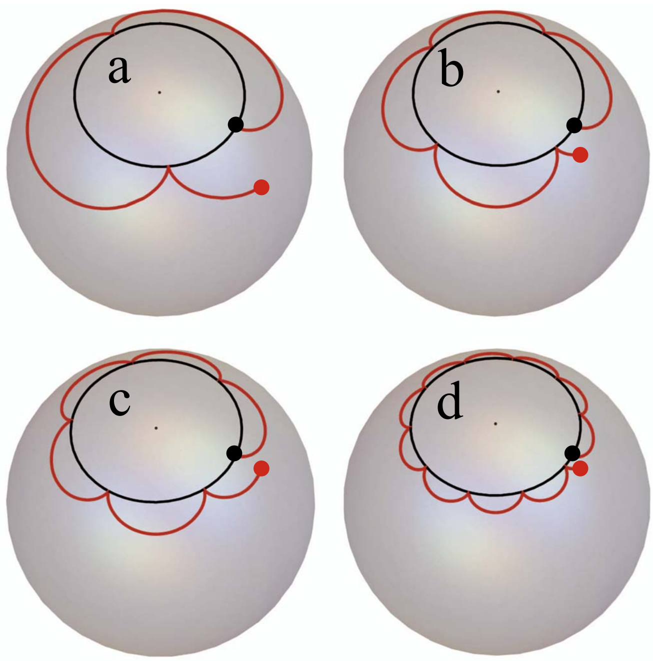


**Figure 3.** Red curves: tracks of the spin expectation (3.2) for $\theta = \pi/6$ and (a) $T = 10$, (b) $T = 16$, (c) $T = 20$, (d) $T = 35$. Black curves: tracks of the driving field (2.2). The black dots indicate the initial spin and the red dots indicate the final spin.

**Table 1.** Values of $T$ times geometrical and dynamical phases, for $T = 30$.

| $\theta$ | $\frac{\pi}{8}$ | $\frac{\pi}{8}$ | $\frac{\pi}{2}$ | $\frac{5\pi}{8}$ |
|---|---|---|---|---|
| $T\gamma_{+,\mathrm{geom}} = -T\pi(1 - \cos\theta)$ | $-7.17$ | $-27.60$ | $-94.25$ | $-130.32$ |
| $T\gamma_{+,\mathrm{dyn}} = -T^2$ | $-900$ | $-900$ | $-900$ | $-900$ |

the same origin as the oscillatory phase corrections, namely transitions between the instantaneous eigenstates (2.5). The final value of the spin gets closer to the initial value as $T$ increases. The track of the smooth part of the evolution would be a circle slightly wider than the track of the driving field $\boldsymbol{B}(t)$, following the average of the nutations (as in figure A1 in appendix B, which illustrates a case with no nutations, i.e. where there are no oscillatory corrections).

## 4. Concluding remarks

The simple corrections reported here contrast with two other kinds of geometric phase corrections.

In the first [16], $H(t)$ is a smooth function that returns only after infinite time, i.e. $H(-\infty) = H(+\infty)$, with $1/T$ representing the slowness with which $H$ is changed. In this case, the series of smooth phase corrections is divergent; the nonanalytic corrections arise from resumming the divergent tail of the series, in a way now standard in asymptotics (e.g. in WKB theory) [18–23], and are exponentially small in $T$, rather than ocsillatory. The exponential smallness of the nonanalytic phase correction reflects the exponentially small probability of a nonadiabatic transition to different states as $T \to \infty$.

In the second, the parameters driving the Hamiltonian are regarded as themselves dynamical variables, and the series of phase corrections is related to a series of reaction forces on the dynamics (classical or quantum) of these variables. An example is quantum chemistry, where the phase describes the electron states and the parameters driving it are the positions of the nuclei. Slowness corresponds to the light-heavy ratio electron mass/nucleus mass. The leading-order reaction force is the energy of the electron eigenstates for frozen nuclear positions, acting as a potential for the nuclear motion; this is the Born–Oppenheimer approximation [24]. The first correction [25–29] involves the 2-form corresponding to the 1-form in (1.5), namely

$$\mathrm{Im}[\langle \nabla n(\boldsymbol{R})\, | \wedge | \nabla n(\boldsymbol{R}) \rangle], \tag{4.1}$$

in which $\boldsymbol{R}$ represents the nuclear coordinates. This gives a force of magnetic type ('geometric magnetism'), depending on the velocity of the nuclei, and reveals a dual role of the 2-form, because this also represents the geometric phase as its flux through a circuit of the electronic Hamiltonian. Geometric magnetism generates sideways displacements when there are spin–orbit interactions, for example in optics [30], where the phenomenon is variously known as the optical Hall effect, the spin–orbit effect of light, or the optical Magnus effect.

Higher-order corrections to the reaction forces are less well understood. A few are known [31–34], and calculations of many reactions for a classical spin model, in which $\boldsymbol{R}$ is a position vector [35], strongly suggests that the series diverges, and that nonanalytic corrections are exponentially weak (see also [36]). But a full understanding of the sequence of reaction forces—an all-orders separation of fast and slow variables—is lacking.

Finally, some comments about the AA phase [15], in which the state, rather than the Hamiltonian, is cycled exactly. Usually, the AA phase is regarded as more general than the $H$-cycled phase: because it is not necessary to cycle the Hamiltonian; because the cycling of the state need not be slow; and because the geometric contribution to the phase depends only on the path of the state in projective Hilbert space, in which the phase is factored out. If there is a slowness parameter, AA provides a natural setting for exploring general adiabatic approximation schemes [11]. But there is also a sense in which AA is less general than $H$-cycling, because the associated adiabatic expansions lack the additional asymptotic richness identified here, of the separation of contributions into powers of slowness, and powers decorated with essential singularities, for example oscillatory. That is why I have relegated the AA phase to a brief description in the appendix. (A separate question concerning the AA phase is what driving Hamiltonian can cause states to cycle exactly; there are various explicit procedures by which any specified state evolution can be generated, as described elsewhere, for example in [37–39].)

## Acknowledgments

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## Data availability statement

No new data were created or analysed in this study.

## Appendix A. Solution of Schrödinger equation

Following the rotating wave procedure of Rabi [12, 13], we transform the evolving state $|\psi(t)\rangle$ to a new state $|\psi_1(t)\rangle$, using a unitary operator $U(t)$ that freezes the rotating magnetic field (2.2), i.e.

$$|\psi(t)\rangle = U(t)\,|\psi_1(t)\rangle \tag{A.1}$$

This operator, generating a rotation of $2\pi t/T$ about the 3-axis, involves the third Pauli matrix $\sigma_3$ in (2.3). Its explicit form, for this effectively spin-1/2 system, is

$$U(t) = \exp\left(-i2\pi\frac{t}{T}\left(\tfrac{1}{2}\sigma_3\right)\right) = \begin{pmatrix} \exp\left(-i\pi\frac{t}{T}\right) & 0 \\ 0 & \exp\left(i\pi\frac{t}{T}\right) \end{pmatrix}. \tag{A.2}$$

The frozen, i.e. time-independent, transformed $H(t)$ is

$$U^\dagger(t)H(t)U(t) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{A.3}$$

This transformed Hamiltonian is not the Hamiltonian $H_1(t)$ that generates the evolution of $|\psi_1(t)\rangle$, because the time-dependence of $U(t)$ must also be incorporated. It follows from (1.1) that

$$i\partial_t|\psi_1(t)\rangle = H_1\,|\psi_1(t)\rangle \tag{A.4}$$

where

$$\begin{aligned} H_1 &= U^\dagger(t)H(t)U(t) - iU^\dagger(t)\partial_t U(t) \\ &= \begin{pmatrix} \cos\theta - \frac{\pi}{T} & \sin\theta \\ \sin\theta & -\cos\theta + \frac{\pi}{T} \end{pmatrix}. \end{aligned} \tag{A.5}$$

The terms involving $\pi$, arising from $\partial_t U$, are the origin of the terms involving $\pi$ in the function $R(T)$ in (2.9). $H_1$ is time-independent, showing that the transformation has successfully frozen the Hamiltonian. This is a special feature of (2.1); it does not happen for general Hamiltonians, which require an iterated sequence of unitary transformations [16].

To solve (A.4), we use the eigenvalues and eigenvectors of the $2\times 2$ Hamiltonian $H_1$. The eigenvalues are

$$H_1\,|\pm_1\rangle = \lambda_\pm|\pm_1\rangle,\ \text{where}\lambda_\pm = \pm\frac{R(T)}{T}, \tag{A.6}$$

where $R(T)$ is given by (2.9). The eigenvectors are

$$|\pm_1\rangle = N_\pm\begin{pmatrix} \pm(T\cos\theta - \pi) + R(T) \\ \pm T\sin\theta \end{pmatrix}, \tag{A.7}$$

where the normalisation constants are

$$N_\pm = \frac{1}{\sqrt{2(R(T)(R(T) \pm (T\cos\theta - \pi)))}}. \tag{A.8}$$

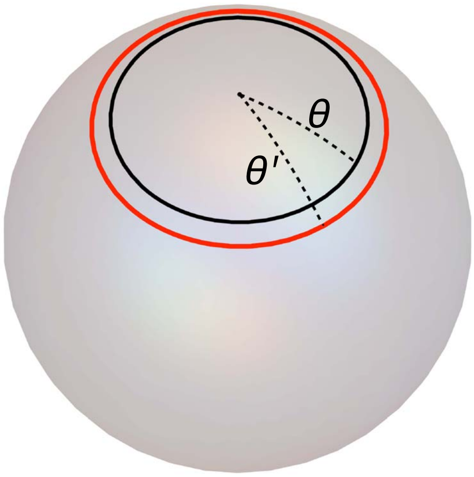


**Figure A1.** Track of spin expectation vector (red curve) for the cyclic state ($A$.5), for $\theta = \pi/6$, $T = 20$, clinging close to the driving field (black curve).

Now we can write the evolution operator $M(t)$ for the initial state $|\psi(0)\rangle$:

$$M(t) = U(t)(\exp(-it\lambda_+)|+_1\rangle\langle +_1| + \exp(-it\lambda_-)|-_1\rangle\langle -_1|). \tag{A.9}$$

A calculation confirms that operating with $M(t)$ on the initial state (2.7) gives the evolution (2.8).

In appendix B we will need the evolution operator over the whole cycle, namely $M(T)$. For any initial state, this relates the state at $t = T$ to the state at $t = 0$ by

$$|\psi(T)\rangle = M(T)|\psi(0)\rangle, \tag{A.10}$$

The explicit form, from (A.9), is

$$M(T) = \begin{pmatrix} M_{11}(T) & M_{12}(T) \\ M_{21}(T) & -M_{22}(T) \end{pmatrix}, \tag{A.11}$$

where the matrix elements are

$$\begin{aligned} M_{11}(T) &= -\cos(R(T)) - i\frac{(\pi - T\cos\theta)\sin R(T)}{R(T)} \\ M_{12}(T) = M_{21}(T) &= i\frac{T\sin\theta\sin R(T)}{R(T)} \\ M_{22}(T) &= -\cos(R(T)) + i\frac{(\pi - T\cos\theta)\sin R(T)}{R(T)}. \end{aligned} \tag{A.12}$$

The eigenvalues of $M$ are

$$\exp(\mp(\pi + R(T))). \tag{A.13}$$

This establishes the significance of the function $R(T)$ in (2.9): it is the phase accumulated by the states that propagate cyclically, in the sense that up to a phase they return at $t = T$ exactly (up to a sign) for any cycle speed, rather than approximately adiabatically. These cyclic states are the eigenvectors of $M$; the one with eigenphase $(\pi - R(T))$ is

$$|\psi_c(t)\rangle = N\exp\left(-i\frac{t}{T}R(T)\right)\begin{pmatrix}\exp\left(-i\pi\frac{t}{T}\right)\left(\cos\theta - \frac{(\pi - R(T))}{T}\right)\\ \exp\left(i\pi\frac{t}{T}\right)\sin\theta\end{pmatrix}, \tag{A.14}$$

with normalisation

$$N = \frac{T}{2(R(T)(R(T) + (T\cos\theta - \pi)))} \tag{A.15}$$

(see (A.7) and (A.8)).

## Appendix B. Connection to AA phase

Cyclic states, that return exactly, irrespective of the Hamiltonian, (which in the present case also returns), are the basis of the AA version of the geometric phase [15]. This is the phase accumulated during the cycle, after subtraction of the dynamical phase, which is defined as the time integral of the instantaneous expectation value of the energy:

$$\gamma_{AA}(T) = \arg\langle\psi_c(0)|\psi_c(T)\rangle + \int_0^T \mathrm{d}t\,\langle\psi_c(t)|H(t)|\,\psi_c(t)\rangle. \tag{B.1}$$

For the cyclic state (A.14), the AA phase is

$$\gamma_{AA}(T) = -\pi\left(1 - \frac{T}{R(T)}\cos\theta\right) + R - \frac{T^2}{R(T)} = -\pi(1 - \cos\theta'(T)), \tag{B.2}$$

where

$$\cos\theta' = \frac{T\cos\theta - \pi}{R(T)} \tag{B.3}$$

The series in $1/T$ is a convergent expansion (see (2.14) whose $T \to \infty$ limit is the standard adiabatic phase (2.6). There are no oscillatory contributions because for cyclic states there are no transitions, an aspect emphasised in a more general study [11] of AA geometric phase corrections.

Figure A1 shows the track of the spin expectation for the cyclic state. By contrast to the noncyclic tracks (see figure 3), these precess on the unit sphere without nutating, describing a cone with polar angle $\theta'(T)$, which gets closer to the driving field cone $\theta$ as $T$ increases. The AA phase $\gamma_{AA}(T)$ in (B.2) is half the solid angle of the spin-traversed cone (B.3) of the cyclic state (A.14).

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