Chaos, Quantum, Number

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Abstract. The three words of the title are connected conceptually, sometimes in unexpected ways

1. Chaos

The following two statements will be deconstructed:
Classical physics is not deterministic.
Classical mathematics is deterministic.
Consider a finite collection of objects interacting through Newtonian forces. If they are isolated, and the forces known, then the future of the system follows inexorably from the starting conditions. This is mathematical determinism: uniqueness of solutions of the relevant differential equations.

The history of the objects may be weakly dependent on the precision with which the initial conditions are specified. This is regular motion, exemplified by the orbits of the planets, or a pendulum swinging under gravity (figures 1(a,b)), or a ball in a rectangular or circular billiard table. Or it may depend sensitively on the initial conditions, as in a pinball machine, or a pendulum whose bob is a magnet, swinging while repelled by magnets on the base (figures 1(c,d)), or a ball on a billiard table shaped like a stadium. This is chaotic motion.

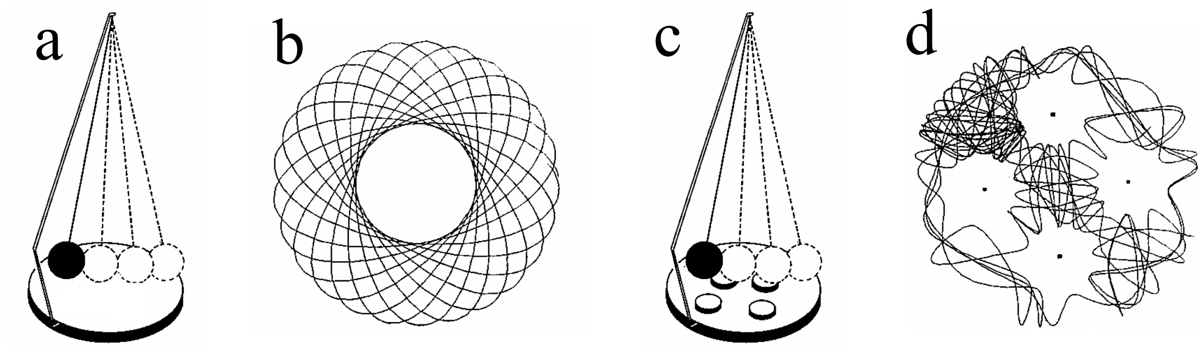

Figure 1. Conical pendulum, swinging (a) under gravity, (c) also with repelling magnets. (b), (d): orbits corresponding to (a), (c).

Mathematical determinism assumes the system is isolated. This contradicts physics, because there is one force that cannot be screened out: gravity. Often, this hardly matters. The moon and the planets are affected by gravity from the stars, but it is not necessary to take this into account when predicting eclipses or programming spacecraft. But sometimes it does matter, as exemplified by an old calculation [1] of gas molecules inside a box, bouncing off each other (I envisage them as classical particles). They are influenced not only by each other and the walls of the box but by gravity outside. The weakest such disturbance is the gravity of a single electron at the observable limit of the universe. Its force will affect the gas molecules differently, and the effect is unpredictable because we don't know where the electron is. After how long will this make the direction of a molecule uncertain by 90°? The answer is: after about 50 collisions - a few nanoseconds. The time is so short because molecules are like convex bodies, whose bounces are unstable, magnifying at each collision. A similar calculation shows that the gravity of billiards players makes the motion of billiard balls uncertain after about ten collisions.

The faraway electron, and the proximate billiards players, make Newtonian mechanics unpredictable in any realistic manner for these unstable systems. But unpredictability is not the same as determinism, as envisaged by Laplace: in principle, the future of the universe could be determined by an intelligent being who (which?) knows the state of all its particles.

But 'in principle' are weasel words. Assuming that mathematical determinism holds for a possibly infinite universe, and that the intelligent being is embodied – we are considering physics, after all – the future of the gas particles or billiard balls would also be sensitive to the gravity of particles in its (his? her? their?) head, or in the computers performing the calculation. Laplacian determinism ignores this self-reference, which sabotages the calculation. The conclusion is that classical physics – as distinct from classical mathematics – is not deterministic in any reasonable sense.

2. Quantum

2.1. Discordance [2]

The physics of the microscopic world could hardly look more different. Instead of Newton's equations, quantum objects (atoms, electrons, nuclei…)

• possess discrete energy levels;
• are indeterministic in a different way from classical chaos (there is no evidence that the decay of a radioactive nucleus originates in the gravity of objects outside it;
• are described in terms of waves as well as particles;
• cannot evolve chaotically if isolated; this is 'quantum suppression of chaos', a mathematical consequence of the discreteness of the energy spectrum;
• are influenced by a nonclassical quantity with the physical dimensions of angular momentum: Planck's constant: h=6.62607014x10^{-34} joule-seconds.

The connection with classical physics is commonly expressed by the Correspondence Principle: quantum objects become classical when h is zero. Physically, this means: when h is negligible in comparison with classical quantities with the same physical dimensions. Although the principle is sometimes useful, and was a historically important guide during the creation of quantum theory, it conceals more than it reveals.

2.2. Singular limits [3, 4]

Quantum-to-classical is an important example of the much more general question of how scientific descriptions at different levels [5], whose formulations appear discordant, are related. How does statistical mechanics 'reduce to' thermodynamics, as the number of particles becomes infinite (described by Philip Anderson as 'More is different') [6, 7]? How does friction-dominated flow reduce to slippery flow as the viscosity vanishes (the obstruction is the unexpected phenomenon of turbulence)? How does wave optics reduce to ray optics as the wavelength gets smaller? Are these apparently incompatible descriptions 'Nonoverlapping Magisteria' (hijacking a phrase used by Stephen Jay Gould)? Or can we achieve 'Consilience' (adapting the lovely old English word revived by E O Wilson)?

Connections between physical descriptions at different levels involve mathematical limits. A central ingredient in understanding the discordance is that in every case the limits are *mathematically singular*. To illustrate the meaning of singular limits in a nontechnincal way [4], imagine biting into an apple and seeing a maggot (figure 2). You are unhappy. If you see half a maggot, you are more unhappy; a quarter of a maggot, more unhappy still; and so on … If the limit were straightforward, encountering no maggot at all should make you infinitely unhappy. Of course it doesn't: the limit of ever-smaller maggot fractions is no unhappiness, not infinite unhappiness. It is a singular limit.

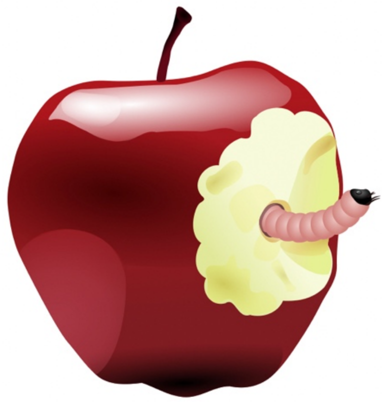

Figure 2. A maggot in the apple

The singular limits perspective enables two distinct insights. The first is *reassurance*: showing how the old theory (classical mechanics) emerges from the deeper one (quantum physics). It must do so – we don't need to solve the Schrödinger equation when programming a spacecraft to reach the Moon. But the limit is often not straightforward. The second insight is *creative*: identifying qualitatively new physics inhabiting the borderlands between the theories.

2.2. Reassurance 1: two beams, and 1+1=2 restored [8])

Imagine two beams of particles (low intensity so they don't collide), or beams of light, oppositely directed towards each other. When they overlap, their intensity, in classical mechanics or ray optics, doubles: 1+1=2. But in quantum physics, the particle beams are waves, and the beams of light are also waves. When waves encounter each other, their intensities do not add. Instead, the waves interfere: constructively (crests) or destructively (troughs), and analysis shows that the intensity of the combined beams takes values between 0 and 4. For intensities, $1+1\neq2$. Don't be surprised by this. As pure mathematics, 1+1=2 is an undisputed theorem. But here we are dealing with applied mathematics, and the theorem may be inapplicable: when two people generate a child, 1+1=3; when two raindrops slide down a car windscreen and coalesce, 1+1=1.

What is the classical, or short-wave, limit of these combined beams? As the wavelength gets smaller (equivalent to diminishing h in the quantum case), the interference oscillations get faster, and the strict mathematical limit would be an intensity taking all values between 0 and 4 in any infinitesimal interval. We never see this, because eventually our instruments cannot resolve the fine oscillations, or they are destroyed by uncontrollable external influences destroying the delicate interferences. This is called decoherence. It was understood qualitatively by Thomas Young in the early 1800s, as the reason why we don't see interference oscillations every time light beams cross. For our two beams, decoherence replaces the oscillating intensity by its average, which simple mathematics reassures us is 2.

2.3. Reassurance 2: Hyperion, and chaos restored [9]

When there is chaos, quantum states get exquisitely sensitive to decoherence as the classical limit is approached. To illustrate this, consider Hyperion, a potato-shaped satellite of Saturn about 300 km across. The rotation of Hyperion is chaotic; an irregular tumbling with initial uncertainty doubling about every 20 days – probably the result of competition between the gravitational effects of Saturn and its largest moon Titan.

Now consider Hyperion as a quantum rotator. Its angular momentum is about 10^{60} h units. How long would it take for quantum physics to supervene, and suppress the chaos? A calculation gives the unexpectedly short time of 40 years. This effect is never observed, because the Hyperion-Saturn-Titan system is not isolated. One of its weakest uncontrolled external influences is a single photon from the Sun, whose re-emission enables us to see Hyperion. Its energy, about 10^{-19} joules, is negligible in comparison with the rotational energy of Hyperion, but 10^{20} times bigger than the separation of its quantum energy levels. The patter of these photons from the faint sunlight reaching Hyperion destroys the delicate quantum coherence in about 10^{-50} seconds: the quantum suppression of chaos is itself suppressed by decoherence, restoring Hyperion's chaos. Reassurance again.

2.4. Creative discordance: statistics of energy levels [10]

Discrete energy levels are intrinsically quantum, and the high-lying levels approach the limit where classical physics should apply. The levels get closer for smaller h; if we magnify to make their mean spacing unity, the study of their arrangement is one aspect of 'semiclassical physics'. A remarkable universality emerges: the statistics of the levels that lie close to each other depends only on the chaology of the system's classical counterpart, not on its details.

The simplest statistic is the probability distribution $P(S)$ of the spacings S between neighbouring levels. This can be illustrated with 'billiard' models: particles moving inside two-dimensional enclosures and bouncing of their walls. As described earlier, the classical regularity or chaos depends on the shape of the enclosure. In the quantum counterpart, the energies are analogous to the frequencies of waves in the enclosure, analogous to vibrations of a drum.

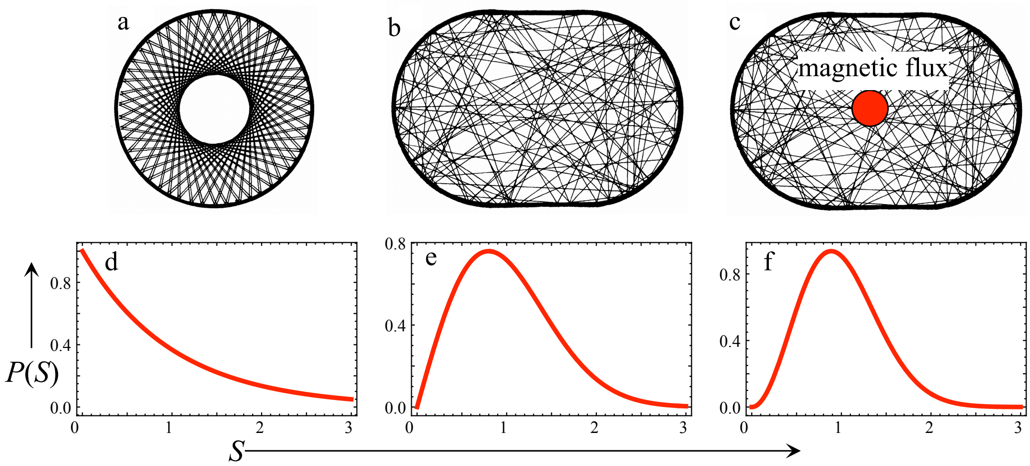

Figure 3. (a-c) orbits in billiards; (a) circular; (b) stadium; (c) stadium with magnetic flux. (d-f) energy-level spacings distributions $P(S)$ corresponding to (a-c).

The first universality class is regular motion. An example is the circular billiard (figure 3a). The spacing distribution (figure 3d) decays from its maximum at S=0: the most probable spacing is zero. This distribution is the same as for uncorrelated random events – apparently counterintuitively, but originating in the fact that the radial and angular quantum aspects give rise to independent overlapping quantum spectra. The second universality class is chaotic motion. An example is the stadium billiard (figure 3b). The structure of the spectrum is very different: instead of separate spectra overlapping randomly, the levels repel each other. The spacings distribution (figure 3e) rises from zero at S=0 to a maximum, and then decays. The third universality class also corresponds to classical orbits that are chaotic, but that are not time-reversible. An example is the stadium billiard with a perpendicular magnetic flux (figure 3c). Again the levels repel each other, but more strongly than before (figure 3f): $P(S)$ rises from zero quadratically, rather than linearly.

The two chaotic universalities correspond to a mathematical structure called *random-matrix theory* [11]. This was originally developed to understand the energy levels of large atomic nuclei, whose many constituent protons and neutrons cause the randomness. (There is a third chaotic universality class, with even stronger level repulsion, corresponding to identical particles obeying the Pauli exclusion principle.)

The spectral universality of high-lying levels is limited. It applies to the statistics of levels close to each other, for example nearest neighbours, as in the examples above. Correlations between more widely spaced energy levels of classically chaotic systems deviate strikingly from those predicted by random-matrix theory. The deviations are associated with the short self-retracing (periodic) classical orbit embedded in the chaos; these are not universal: they vary from system to system.

The quantum chaology of energy-level correlations is an example of rich physics inhabiting the borderland between the classical and quantum worlds. It is a consequence of the singular limit connecting the two theories.

3. Number [12]

F C Frank (unpublished, 1976) stated: "Physics is not just Concerning the Nature of Things, but Concerning the Interconnectedness of all the Natures of Things." An 'interconnectedness' apparently completely unrelated to quantum chaology concerns the arrangement of the prime numbers: 2,3,5,7,11,13,17,19, 23…Represented as a staircase, the number of primes less than x is an erratic sequence of steps (figure 4a). On a larger scale, the steps are invisible (figure 4b), and the 'staircase' appears as a smooth hillside (whose mathematical form is known). But magnification reveals that the steps are still there, apparently irregularly located (figure 4c).

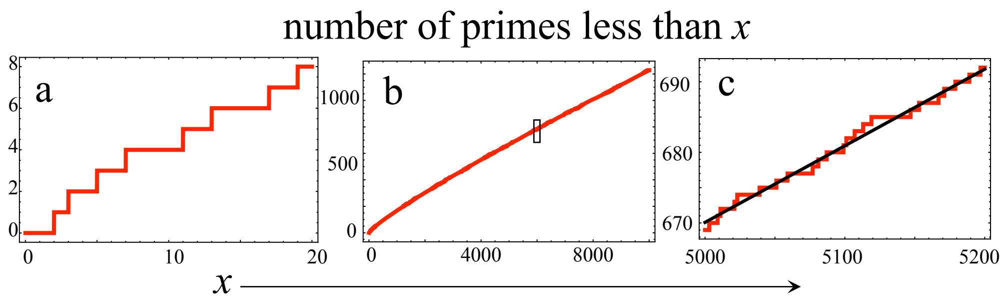

Figure 4. Staircase of the primes. (a) Showing steps; (b) zooming out, steps invisible; (c) magnifying the little box in (b) to see the steps.

The irregularity of the primes is captured by a different set of numbers, t_n, representing 'frequencies' when the fluctuations (staircase minus hillside) are represented as a series of oscillations. The first few are t_1=14.1347, t_2=21.022, t_3=25.0109, t_4=30.4249… The numbers t_n are the Riemann zeros. An important unsolved problem in mathematics, with implications in many areas, including the statistics of the fluctuations of the primes, is to prove the truth or falsity of the *Riemann hypothesis (RH)*. This states that all the t_n are real numbers. It is known that the first ten trillion are real, infinitely many are real, and at least 40% are real. But this is not a proof, And settling RH has resisted intense efforts over more than 150 years.

The unexpected 'interconnection' is that the interpretation of the Riemann zeros as frequencies is not accidental. A mathematical analogy between quantum chaology (including random-matrix theory), and formulas involving the Riemann zeros, hints strongly that these numbers can be interpreted as the vibration frequencies, or, equivalently, energy levels, of a quantum system in the third universality class of quantum chaology. The nature of this quantum system is unknown, but the analogy enables several properties to be identified, such as: the system has a classical counterpart, which is chaotic, lacks time-reversal symmetry, and possesses periodic orbits whose periods are multiple of logarithms of primes.

These identifications suffice to predict statistics of the Riemann zeros, assuming they are real. When tested against computations of billions of the Riemann zeros t_n near n=10^{23}, the analogy is spectacularly successful, far beyond random-matrix theory. The distribution of spacings of neighbouring zeros is indistinguishable from the random-matrix prediction (figure 3f); and for wider separations of zeros the statistics deviate from those of random-matrix theory in precisely the same way as their quantum chaology counterparts (the deviations depend on small primes, i.e. short periodic orbits.

Although this seems promising, the persisting failure to prove the RH counsels caution. The basis of the analogy is a 'known unknown': the chaotic classical dynamical system. But the possibility remains that RH could be false; there are some slight inconsistencies in the analogy, hinting at a lurking 'unknown unknown' that could ruin it. As Piet Hein declared: "Problems worthy of attack, Prove their worth by hitting back"

4. Concluding remarks

- Classical physics (chaotic or not) is very different from quantum.
- The discordance is associated with the fact that the classical limit $h \to 0$ is singular.
- Singular limits complicate the reassurance that classical physics must emerge when h is negligible.
- Singular limits also point to new physics inhabiting the borderlands between the classical and quantum worlds.
- A bonus is the analogy between quantum chaology and the prime numbers.

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