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A Geometric- Phase Timeline

Global change
without local
change—a
connecting idea
in the physics of
optical, quantum
and other waves—
has a multistranded
history spanning
two centuries.



A view near Skreen, in the west of Ireland, through a thin crystal sandwiched between crossed polarizers. The dark band across the interference fringes corresponds to a geometric phase of π .

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Waves are central to our understanding of the physical world, and phase is the feature that distinguishes waves from classical particles. As waves evolve, their phases change, at rates that can themselves change when conditions change. In light, refracting materials or polarizers can change the phase; electromagnetic fields can change the phase of quantum charged particles; changing the depth of water can change the phase of waves on it.

It seems natural to calculate the total phase change at the end of a process by adding up all the local phase changes—calculating the optical path length or, more generally, integrating the instantaneous frequency. This “dynamical phase” can be regarded as a system’s partial answer to such

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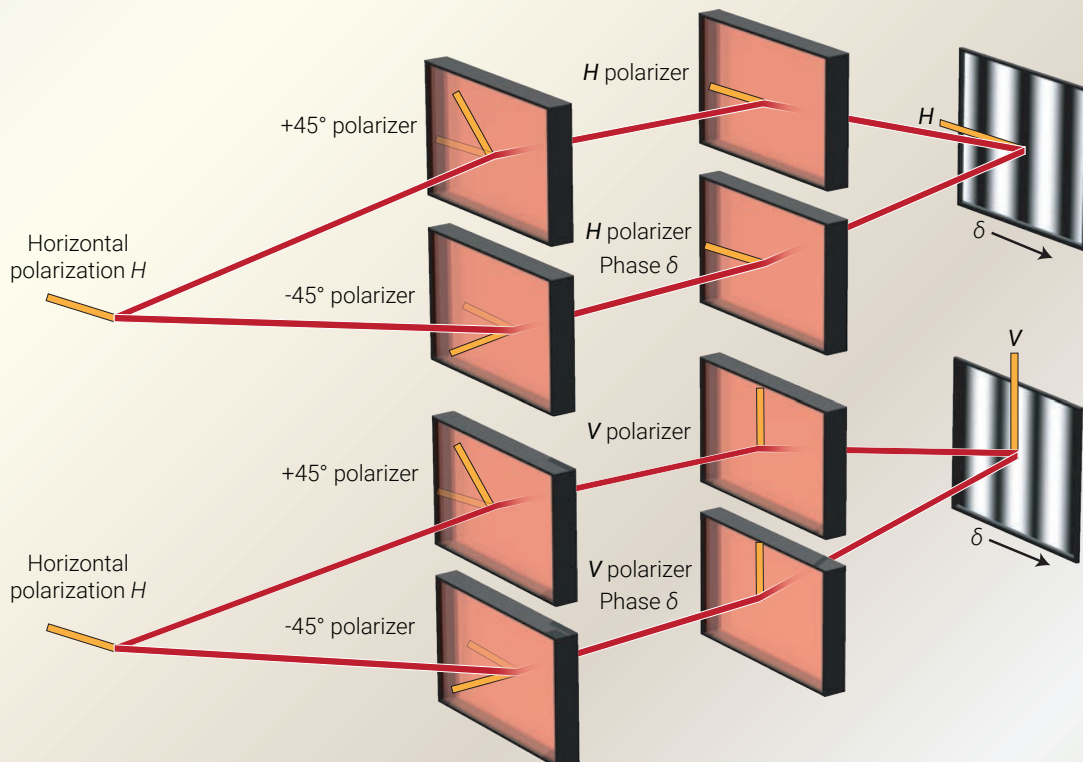
questions as “How far, or for how long, have you traveled?” But in many cases this answer is incomplete. There is an additional, geometric contribution to the phase—geometric in the sense of depending only on

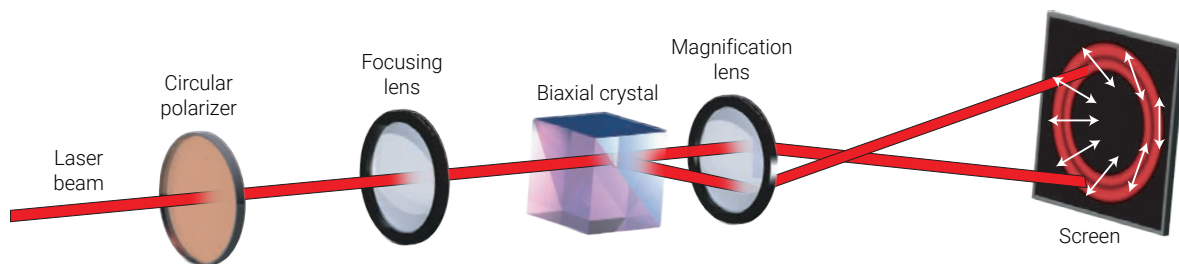
the sequence of changed conditions, not on how fast the changes are made. It can be regarded as the system’s answer to “What changes have you experienced?” or “Where have you been?”

Elsewhere I have described how I found this geometric phase in 1983 (reported in a paper published 40 years ago this month, in 1984), and how I subsequently unearthed partial anticipations of the idea. Here I will not give a detailed exposition of the theory underlying geometric phases or review the substantial theoretical and experimental literature

about geometric phases in optics; others have done that. My aim is limited: to give my current, personal

A schematic of Fresnel–Arago experiments underlying their fifth law of polarized-light interference, with fringes relatively shifted by π .





A schematic of a conical-refraction experiment with M. Jeffrey. Laser light passes a circular polarizer, focused by a lens onto a biaxial crystal ($\text{KGd}(\text{WO}_4)_2$) and magnified by a second lens onto a screen. The polarization rotation around the Hamilton–Lloyd rings exhibits the geometric phase π .

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perspective by revisiting the history, pre- and post-1983, including important early discoveries.

Early phases in polarization optics

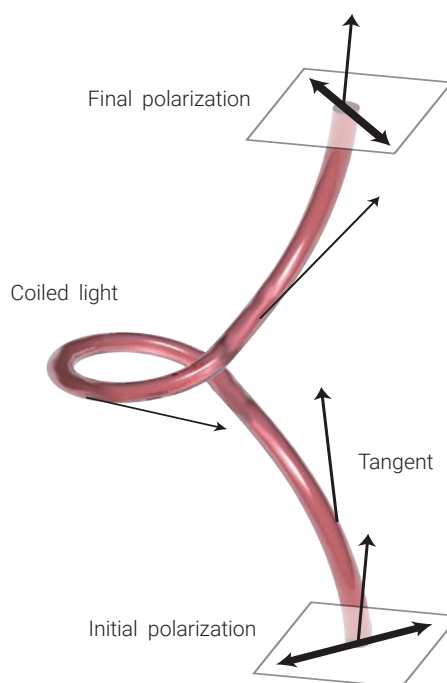
Following Thomas Young’s discovery of the interference of light waves, detailed experimental and theoretical studies by Augustin-Jean Fresnel and Dominique François Jean Arago culminated in 1819 in their five laws governing the interference of polarized light. As was pointed out recently, the fifth law, almost never mentioned in textbooks, can be regarded as the first notion of geometric phase.

It can be described by comparing two experiments, following Oriol Arteaga. In both, a beam of horizontally polarized light is split into two beams, polarized at $+45^\circ$ and -45° , with the -45° beam phase-shifted by δ . The difference is in the interference generated by subsequent analysis to a common polarization. In the first experiment, the common polarization is horizontal, and the intensity of their interference is proportional to $\cos^2(\frac{1}{2}\delta)$. In the second experiment, the common polarization is vertical, and the intensity of their interference is proportional to $\sin^2(\frac{1}{2}\delta)$. There is a relative phase shift of 180° between the interference fringes in the two experiments; as will be explained later, this corresponds to a geometric phase of π .

On to the early 1830s, and William Rowan Hamilton’s prediction of conical refraction: Light, passing through a slab of biaxially refracting crystal cut perpendicular to one of its optical axes, would emerge as a hollow cylinder, visible as a bright ring on a screen. Its observation by Humphrey Lloyd created a sensation and made Hamilton instantly famous. Lloyd noticed that the ring system—soon resolved into two rings—is linearly polarized, with

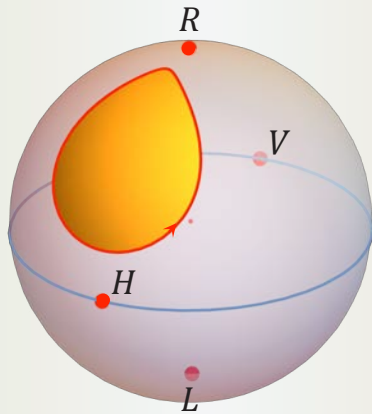
the polarization direction changing by 180° around the ring; as will be explained later, this too corresponds to a geometric phase of π .

Fast-forward to 1926, and the geometer Ettore Bortolotti’s identification of a feature that would become central to geometric phases. This was that Maxwell’s equations imply that the electric and magnetic fields of light are parallel-transported as its direction changes. The observation was an application of mathematics discovered by Carl Friedrich

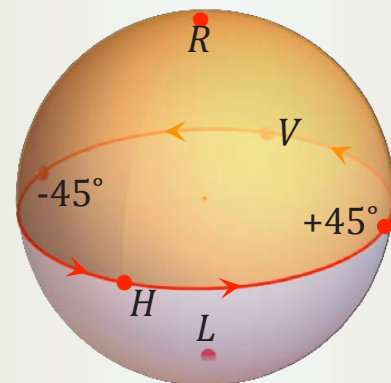
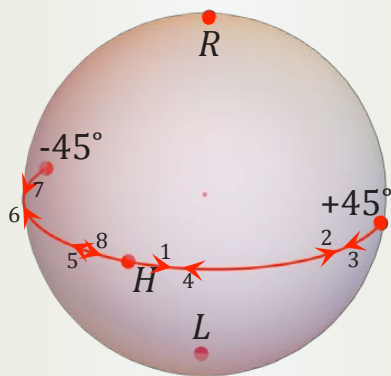


Vladimirsky predicted that the polarization of a twisted light beam would turn by the solid angle through which the tangent has rotated—a prediction later verified by A. Tomita and R. Chiao with laser light in an optical fiber.

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Pancharatnam's law: Polarization transported around a circuit on the Poincaré sphere acquires a geometric phase of half the circuit's solid angle (R, L, H and V represent right- and left-circular and horizontal- and vertical-linear polarizations).



Explanation of the Fresnel–Arago π geometric phase by Pancharatnam's half-solid-angle on the Poincaré sphere, following O. Arteaga. Top: First experiment, where the phase difference is equivalent to the self-retracing circuit $H/+45^\circ/H/-45^\circ/H$ (i.e., 1–2–3–4–5–6–7–8–1 on the sphere), with solid angle zero. Bottom: Second experiment, where the phase difference is equivalent to the circuit $H/+45^\circ/V/-45^\circ/H$, with solid angle of a hemisphere (yellow), i.e., 2π .

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Gauss a century earlier, concerning vectors transported around circuits on curved surfaces: Even if the vector is never twisted about the local surface normal, it will return pointing away from its starting direction—global change without local change. If the surface is a sphere, the direction has turned by the solid angle subtended by the circuit at the center.

The parallel transport of electromagnetic fields was rediscovered in 1938 by Sergei Mikhailovich Rytov, prompting his student Vassily Vladimirsky to make a prediction in 1941. If a beam of polarized light is twisted so that its direction at the end is the same as at the start, its polarization direction will have turned by the solid angle through which the beam's tangent has turned. In 1986, Vladimirsky's classical-optics phenomenon was observed in laser light in a coiled single-mode optical fiber and reinterpreted as the quantum geometric phase for a stream of spin-1 photons (see figure, p. 45).

Pancharatnam's contribution

It's 1956. Enter Sivaramakrishna Pancharatnam, a brilliant nephew of the optics Nobelist Chandrasekhara Venkata Raman. Revisiting the work of Fresnel and Arago (though apparently unaware of their fifth law), Pancharatnam made a seminal contribution, unappreciated for three decades, to our understanding of polarized-light interference.

Pancharatnam defined the phase difference between two polarization states, A and B , as the phase of their overlap (scalar product). It follows that A and B are "in phase" if the intensity of their superposition $A + B$ is maximal. Importantly, he showed that this relation—Pancharatnam's connection—is nontransitive: If A is in phase with B and B is in phase with C , A need not be in phase with C .

Pancharatnam interpreted polarization states as points on the Poincaré sphere. His central result is equivalent to the following: In a cycle of polarizations $A \rightarrow B \rightarrow C \rightarrow A$, where the legs are geodesics on the sphere, the phase difference between the final and initial states A —the geometric phase—is minus half the solid angle of the spherical triangle ABC on the Poincaré sphere. Pancharatnam's solid-angle result generalizes immediately to a circuit where the polarization changes continuously. His version of geometric phase immediately explains the difference between the Fresnel–Arago experiments underlying their fifth law.

Quantum chemistry and geometric magnetism

At first encounter, the next contribution, in 1958, appears unrelated. Christopher Longuet-Higgins, Maurice Pryce, Uno Öpik and Robert Sack were studying the quantum chemistry of molecules, in particular the Born–Oppenheimer approximation. In that approximation, the heavy nuclei are regarded as frozen while the swift electrons solve their Schrödinger equation, resulting in instantaneous electron states and energies depending parametrically on the nuclear configuration. Longuet-Higgins and colleagues discovered that around nuclear configurations for which the electron energies

are degenerate, the electron wavefunctions change sign. This corresponds to another π geometric phase.

In a fundamental shift of emphasis, they realized that for the total wavefunction of the molecule to be single-valued, this π phase of the electrons must be incorporated as a continuation condition on the Schrödinger equation for the slow nuclei—thereby modifying the molecule’s vibration–rotation levels. Such “geometric reactions” would become a major theme in geometric-phase-related research.

A year later came another apparently unrelated development. Yakir Aharonov and David Bohm calculated the scattering of electrons from a line of magnetic flux from which they are shielded. They discovered that even though the electrons feel no magnetic or electric field, they experience a phase shift from the inaccessible flux. This Aharonov–Bohm (AB) effect is now understood as a geometric phase of electrons circling the flux.

In a major development in 1979, Chester Alden Mead and Donald Truhlar generalized the π phase of Longuet-Higgins et al. Mead and Truhlar discovered that the reaction of the geometric phase of the fast electrons on the slow nuclei that drive them is a velocity-dependent force of magnetic type. They described this “geometric magnetism” as the “molecular Aharonov–Bohm effect.”

Pancharatnam’s version of geometric phase immediately explains the difference between the Fresnel–Arago experiments underlying their fifth law.

The general geometric phase emerges

When I wrote my paper identifying the geometric phase in 1983, I was unaware of all previous work except the AB effect. My interest was in quantum waves (and by implication more general waves) evolving according to the time-dependent Schrödinger equation while driven by a Hamiltonian depending slowly on parameters $\mathbf{R}(t/T) = \{X(t/T), Y(t/T) \dots\}$, with T large, so $1/T$ is a slowness parameter.

The adiabatic theorem of Max Born and Vladimir Fock guarantees that a system starting in an eigenstate of the initial Hamiltonian will cling close to that eigenstate as it slowly evolves, and at the end of the cycle will be in the same state, apart from a phase factor. The dominant contribution is the dynamical phase:

the integral of the instantaneous energy, proportional to T . The leading-order correction is independent of T , and depends only on the geometry of the circuit in \mathbf{R} space, not on the rate at which it is traversed; that is why I called it the geometric phase.

The simplest expression for the phase was as the integral of a quantity, called a connection (mathematically, a 1-form) around the circuit. But more fundamental was the representation, obtained from Stokes’ theorem, inspired by analogy with Faraday’s law: The geometric phase is the flux through the circuit of a quantity, later called curvature (mathematically a 2-form), in \mathbf{R} space. Most important are the points \mathbf{R} where the evolving eigenstate is degenerate; there, the curvature possesses a singularity of monopole type. For a small planar circuit centered on a degeneracy, the geometric phase is π .

My emphasis on circuits arose from the fact that the overall phase for an eigenstate at \mathbf{R} in parameter space can be chosen arbitrarily. This “gauge dependence” is present in the geometric-phase connection but disappears when integrated around a circuit—the geometric phase is gauge-independent. The phase curvature is fundamental because unlike the connection it is gauge-independent everywhere in \mathbf{R} space. In subsequent studies, particularly in polarization optics, gauge-independent geometric phases have

been defined for open paths. These are equivalent to the geometric phases obtained by closing the path, with the final state defined as being in-phase, in Pancharatnam's sense, with the state at the endpoint of the open path.

A simple example was the application to quantum spins S (integer or half-odd integer) in a state n (that is, $n = -S, -S+1, \dots, S$), with parameters $\mathbf{R}(t)$ representing a driving magnetic-field vector. The monopole generates a geometric phase of $-n$ times the solid angle enclosed by the circuit in \mathbf{R} space. The analogy between the $S = 1/2$ case and Pancharatnam's solid-angle formula was soon understood and explored in detail. Although photons are spin-1 particles, transversality implies that a beam's polarization

is represented by the two-state algebra of spin- $1/2$. And the Hamilton-Lloyd π phase was later explained as a degeneracy—at the center of the conical refraction ring in direction space—of a 2×2 matrix originating in Maxwell's equations applied to a biaxial crystal.

From Barry Simon, commenting in 1983 on a preprint of my paper, came two insights. First, the underlying geometry is a physical application of the mathematics of fiber bundle theory, with the phase curvature identified as the curvature of the manifold of parameters and the counting of degeneracies being related to the Chern class. Second, the phase curvature was central to the then-current research on the quantum Hall effect for electrons in solids. This kick-started an explosion of research in topological condensed matter that continues today.

Extending geometric phase

It gradually emerged that the phase curvature appears throughout physics, in many guises. As well as its flux generating the phase, it is the Mead-Truhlar geometric magnetic field generating quantum reaction forces; this reappeared in optics, with geometric magnetism referred to as the "optical Magnus effect," "spin-orbit effect of light" or "optical Hall effect." The reaction of the polarization state on the position variables causes a shift in, for example, a speckle pattern in a multi-mode fiber. The curvature also underlies the density of optical vortices and polarization singularities; it

is the vorticity of the Poynting vector; and it determines nonconservative optical "curl forces" on small dielectric particles.

Developments soon followed. In the basic geometric-phase theory, the state being cycled is nondegenerate. If a cluster of N instantaneous states is degenerate for all parameters, slowly cycling can generate transitions between states within the cluster. Instead of a phase factor, there is a geometric $N \times N$ unitary matrix. This is the "non-Abelian" case described in 1984 by Frank Wilczek and Anthony Zee.

In 1985, John Hannay discovered a counterpart of the geometric phase in the classical dynamics of slowly cycled integrable (that is, nonchaotic) bound systems. Such motion is described

by angle variables. At the end of a cycle, "Hannay's angles" give the geometric (slowness-independent) additional contribution to the accumulated angles. Semiclassical theory establishes the j^{th} Hannay angle as minus the derivative of the geometric phase of the associated state with respect to the j^{th} quantum number. Hannay's angle provides alternative descriptions of the Foucault pendulum and the rotation angle of a precessing spinning-top. The counterpart of the quantum geometric phase for chaotic classical dynamics was formulated with Jonathan Robbins, but it remains to be explored in detail.

Geometric phases are artifacts of separation

In an important study in 1987, Yakir Aharonov and Jeeva Anandan eliminated the slowness condition by calculating the geometric phase for states that are cycled exactly in the Hilbert space of states, rather than approximately in the space of a slowly changed Hamiltonian. I recognized the value of this formulation, but noted the absence of a prescription to generate exact cycling. Much later, I learned that there exist explicit Hamiltonian procedures to drive any prescribed sequence of states, including cyclic, at any speed.

Deeper consideration of the molecular quantum chemistry phase leads to a fundamental general insight. What we call parameters are themselves

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dynamical variables; they cannot act on other parts of a system without themselves being acted on. If the Schrödinger equation for the complete molecule could be solved without parameters being separated, all observable quantum phenomena (energy levels, for instance, and reaction rates) could be calculated holistically, without considering the geometric phase. Therefore the geometric phase is an artifact—a consequence of our decision to separate a system into parts.

So “in principle,” geometric phases can be eliminated by including driving parameters when modeling a system. This does not mean that geometric phases are scientifically redundant, because “in principle” obscures scientific realities. In large molecules, exact quantum calculation, including electrons and nuclei together, is not currently feasible. Even if it became so, it would not provide the explanatory picture supplied by the Born–Oppenheimer approximation, extended to include the geometric phase. Moreover, in many experiments, especially in optics, control of the parameters involves laboratory equipment (such as polarizers and spatial light modulators) on which the reaction of light is undetectably weak as well as uninteresting.

Taken to extremes, the holistic approach would include the entire universe. But science succeeds by deciding what is relevant and neglecting what is not. As Isaac Newton put it: “To explain all nature is too difficult a task for any one man or even for any one

age. ‘Tis better to do a little with certainty & leave the rest for others that come after. ...”

Nevertheless, in circumstances where geometric phases are useful, post-adiabatic approximations can be investigated. In 1987 I calculated phase corrections beyond geometric (of order $1/T$, $1/T^2$, etc.) for the case where the slow driving occurs smoothly over infinite time. This revealed that the theory is asymptotic: The correction terms initially get smaller, but then diverge, reflecting the weak (of order $\exp[-1/T]$) nonadiabatic transitions to other states. This has given rise to a great deal of physical mathematics. Very recently, I explored an exactly solvable case where the driving is a cycle of finite duration T ; the series in powers of $1/T$ converges, but the terms are multiplied by fast oscillations (for example, $\cos T$), reflecting oscillations in the probability of transitions to other states.

Calculating the corresponding series of reaction forces (post-Born–Oppenheimer in the case of molecules, and post-geometric magnetism) is much more difficult. Detailed exploration of a model, with Pragma Shukla, suggests that the series of reaction forces also diverges. This is an important unsolved problem because it bears on a deep question: How strictly can the fast and slow parts of a system be separated? **OPN**

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References and Resources

- ▶ H.C. Longuet-Higgins et al. “Studies of the Jahn-Teller effect II. The dynamical problem,” *Proc. R. Soc. Lond. A* **244**, 1 (1958).
- ▶ M.V. Berry. “Quantal phase factors accompanying adiabatic changes,” *Proc. R. Soc. Lond. A* **392**, 45 (1984).
- ▶ B. Markovski and S.I. Vinitzky (Eds.). *Topological Phases in Quantum Theory*, World Scientific (1989) (includes papers by Rytov and Vladimirkii).
- ▶ A. Shapere and F. Wilczek (Eds.). *Geometric Phases in Physics*, World Scientific (1989) (includes many important early papers).
- ▶ M.V. Berry. “Anticipations of the geometric phase,” *Phys. Today* **43**(12), 34 (1990).
- ▶ M.V. Berry and M.R. Dennis. “Polarization singularities in isotropic random waves,” *Proc. R. Soc. Lond. A* **457**, 141 (2001).
- ▶ M.V. Berry and M.R. Jeffrey. “Conical diffraction: Hamilton’s diabolical point at the heart of crystal optics,” *Prog. Optics* **50**, 13 (2007).
- ▶ M.V. Berry. “Geometric phase memories,” *Nat. Phys.* **6**, 148 (2010).
- ▶ M.V. Berry and P. Shukla. “High-order classical adiabatic reaction forces: Slow manifold for a spin model,” *J. Phys. A* **43**, 045102 (2010).
- ▶ M.V. Berry and P. Shukla. “Physical curl forces: Dipole dynamics near optical vortices,” *J. Phys. A* **46**, 422001 (2013).
- ▶ E. Cohen et al. “Geometric phase from Aharonov–Bohm to Pancharatnam–Berry and beyond,” *Nat. Rev. Phys.* **1**, 437 (2019).
- ▶ O. Arteaga. “Fresnel–Arago fifth law of interference: The first description of a geometric phase in optics,” *J. Mod. Opt.* **68**, 350 (2021).
- ▶ M.V. Berry. “Smooth and oscillatory geometric phase corrections for driven spins,” *Eur. J. Phys.* **44**, 065402 (2023).
- ▶ L. Garza-Soto et al. “Deciphering Pancharatnam’s discovery of geometric phase: Retrospective,” *J. Opt. Soc. Am. A* **40**, 925 (2023).